# Problem 3.12

Find $\Phi(p,t)$ for the free particle in terms of the function $\phi(k)$ introduced in Equation 2.101. Show that for the free particle $|\Phi(p,t)|^2$ is independent of time. *Comment*: the time independence of $|\Phi(p,t)|^2$ for the free particle is a manifestation of momentum conservation in this system.

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### Solution

The general formulas for the Fourier transform of a function $f(x)$ and its corresponding inverse Fourier transform are as follows.

$$\begin{cases} F(k) = \sqrt{\dfrac{|b|}{(2\pi)^{1-a}}} \displaystyle\int_{-\infty}^{\infty} e^{ibkx} f(x)\, dx \\ f(x) = \sqrt{\dfrac{|b|}{(2\pi)^{1+a}}} \displaystyle\int_{-\infty}^{\infty} e^{-ibkx} F(k)\, dk \end{cases}$$

The Fourier transform can be used to solve linear partial differential equations over the whole line. Any choice for $a$ and $b$ is acceptable, and how one chooses to define the Fourier transform really comes down to personal preference. In Chapter 2, for example, the Schrödinger equation was solved using $a = 0$ and $b = -1$.

$$\begin{cases} \mathcal{F}\{\Psi(x,t)\} = \tilde{\Psi}(k,t) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{-ikx} \Psi(x,t)\, dx \\ \mathcal{F}^{-1}\{\tilde{\Psi}(k,t)\} = \Psi(x,t) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{ikx} \tilde{\Psi}(k,t)\, dk \end{cases}$$

One choice for $a$ and $b$ is special in quantum mechanics, though: $a = 0$ and $b = -1/\hbar$.

$$\begin{cases} \mathscr{F}\{\Psi(x,t)\} = \Phi(p,t) = \dfrac{1}{\sqrt{2\pi\hbar}} \displaystyle\int_{-\infty}^{\infty} e^{-ipx/\hbar} \Psi(x,t)\, dx \\ \mathscr{F}^{-1}\{\Phi(p,t)\} = \Psi(x,t) = \dfrac{1}{\sqrt{2\pi\hbar}} \displaystyle\int_{-\infty}^{\infty} e^{ipx/\hbar} \Phi(p,t)\, dp \end{cases}$$

$\Psi(x,t)$ is the position-space wave function because $|\Psi(x,t)|^2$ represents the probability distribution for the particle's position. On the other hand, $\Phi(p,t)$ is the momentum-space wave function because $|\Phi(p,t)|^2$ represents the probability distribution for the particle's momentum. These formulas are a result of solving the eigenvalue problem for the momentum operator.

$$\hat{p} f(x) = p f(x)$$

$$-i\hbar \frac{d}{dx} f(x) = p f(x)$$

$$\frac{df}{dx} = \frac{ip}{\hbar} f(x)$$

$$f(x) = A e^{ipx/\hbar}$$

This is a non-normalizable function, so the spectrum is continuous, meaning the continuous Dirac-analogs of Equations 3.10 and 3.11 on page 93 apply. Since $\hat{p}$ is a hermitian operator, the eigenfunctions associated with the real, distinct eigenvalues are orthogonal.

$$\begin{aligned}
\langle f' \mid f \rangle = \int_{-\infty}^{\infty} (Ae^{ip'x/\hbar})^*(Ae^{ipx/\hbar})\,dx &= \int_{-\infty}^{\infty} (A^*e^{-ip'x/\hbar})(Ae^{ipx/\hbar})\,dx \\
&= |A|^2 \int_{-\infty}^{\infty} e^{i(p-p')x/\hbar}\,dx \\
&= A^2 \left[2\pi\delta\left(\frac{p-p'}{\hbar}\right)\right] \\
&= A^2 \left[2\pi|-\hbar|\delta(p'-p)\right] \\
&= 2\pi\hbar A^2\delta(p'-p)
\end{aligned}$$

Determine $A$ by requiring the magnitude of the delta function to be 1.

$$2\pi\hbar A^2 = 1 \quad \rightarrow \quad A = \frac{1}{\sqrt{2\pi\hbar}}$$

Consequently,

$$f(x) = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}.$$

$\hat{p}$ is a hermitian operator, so any function in position-space, including the one we're most interested in, $\Psi(x,t)$, can be expressed as a linear combination of its eigenfunctions.

$$\Psi(x,t) = \int_{-\infty}^{\infty} B(p,t)\left(\frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}\right)dp$$

By comparing this to the general formulas, we see that this is a very special inverse Fourier transform, one where $a = 0$ and $b = -1/\hbar$. The initial value problem for a free particle is

$$\begin{aligned}
&i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}, \quad -\infty < x < \infty,\ t > 0 \\
&\Psi(x,0) = \Psi_0(x).
\end{aligned}$$

Take the Fourier transform of both sides of each equation to solve it.

$$\begin{aligned}
&\mathcal{F}\left\{i\hbar\frac{\partial\Psi}{\partial t}\right\} = \mathcal{F}\left\{-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}\right\} \\
&\mathcal{F}\{\Psi(x,0)\} = \mathcal{F}\{\Psi_0(x)\}
\end{aligned}$$

Use the fact that the transform is a linear operator.

$$\begin{aligned}
&i\hbar\mathcal{F}\left\{\frac{\partial\Psi}{\partial t}\right\} = -\frac{\hbar^2}{2m}\mathcal{F}\left\{\frac{\partial^2\Psi}{\partial x^2}\right\} \\
&\tilde{\Psi}(k,0) = \tilde{\Psi}_0(k) = \phi(k)
\end{aligned}$$

Transform the derivatives and solve the resulting differential equation for $\tilde{\Psi}(k,t)$.

$$i\hbar\frac{d\tilde{\Psi}}{dt} = -\frac{\hbar^2}{2m}(ik)^2\tilde{\Psi}(k,t)$$

$$\frac{d\tilde{\Psi}}{dt} = -\frac{i\hbar}{2m}k^2\tilde{\Psi}(k,t)$$

$$\tilde{\Psi}(k,t) = C(k)\exp\left(-\frac{i\hbar}{2m}k^2t\right)$$

Use the transformed initial condition to determine $C(k)$.

$$\tilde{\Psi}(k,0) = C(k) = \phi(k)$$

As a result,

$$\tilde{\Psi}(k,t) = \phi(k)\exp\left(-\frac{i\hbar}{2m}k^2t\right).$$

Take the inverse Fourier transform to get $\Psi(x,t)$.

$$\begin{aligned}\Psi(x,t) &= \mathcal{F}^{-1}\{\tilde{\Psi}(k,t)\} \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\phi(k)\exp\left(-\frac{i\hbar}{2m}k^2t\right)dk\end{aligned}$$

Now that the position-space wave function is known, the momentum-space wave function can be found by taking the Fourier transform with $a = 0$ and $b = -1/\hbar$.

$$\begin{aligned}\Phi(p,t) &= \mathscr{F}\{\Psi(x,t)\} \\ &= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\Psi(x,t)\,dx \\ &= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\phi(k)\exp\left(-\frac{i\hbar}{2m}k^2t\right)dk\right]dx \\ &= \frac{1}{2\pi\sqrt{\hbar}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ipx/\hbar}e^{ikx}\phi(k)\exp\left(-\frac{i\hbar}{2m}k^2t\right)dk\,dx \\ &= \frac{1}{2\pi\sqrt{\hbar}}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{i(k-p/\hbar)x}\,dx\right]\phi(k)\exp\left(-\frac{i\hbar}{2m}k^2t\right)dk \\ &= \frac{1}{2\pi\sqrt{\hbar}}\int_{-\infty}^{\infty}\left[2\pi\delta\left(k-\frac{p}{\hbar}\right)\right]\phi(k)\exp\left(-\frac{i\hbar}{2m}k^2t\right)dk \\ &= \frac{1}{\sqrt{\hbar}}\int_{-\infty}^{\infty}\delta\left(k-\frac{p}{\hbar}\right)\phi(k)\exp\left(-\frac{i\hbar}{2m}k^2t\right)dk \\ &= \frac{1}{\sqrt{\hbar}}\phi\left(\frac{p}{\hbar}\right)\exp\left[-\frac{i\hbar}{2m}\left(\frac{p}{\hbar}\right)^2t\right] \\ &= \frac{1}{\sqrt{\hbar}}\phi\left(\frac{p}{\hbar}\right)\exp\left(-\frac{ip^2}{2\hbar m}t\right)\end{aligned}$$

The probability distribution function for the free particle's momentum is then

$$\begin{aligned}
|\Phi(p,t)|^2 &= \Phi^*(p,t)\Phi(p,t) \\
&= \left[\frac{1}{\sqrt{\hbar}}\phi\left(\frac{p}{\hbar}\right)\exp\left(-\frac{ip^2}{2\hbar m}t\right)\right]^* \left[\frac{1}{\sqrt{\hbar}}\phi\left(\frac{p}{\hbar}\right)\exp\left(-\frac{ip^2}{2\hbar m}t\right)\right] \\
&= \left[\frac{1}{\sqrt{\hbar}}\phi^*\left(\frac{p}{\hbar}\right)\exp\left(\frac{ip^2}{2\hbar m}t\right)\right] \left[\frac{1}{\sqrt{\hbar}}\phi\left(\frac{p}{\hbar}\right)\exp\left(-\frac{ip^2}{2\hbar m}t\right)\right] \\
&= \frac{1}{\hbar}\phi^*\left(\frac{p}{\hbar}\right)\phi\left(\frac{p}{\hbar}\right) \\
&= \frac{1}{\hbar}\left|\phi\left(\frac{p}{\hbar}\right)\right|^2,
\end{aligned}$$

which is independent of time.